

INEQUALITIES FOR OPERATOR SPACE NUMERICAL RADIUS OF 2×2 BLOCK MATRICES

MOHAMMAD SAL MOSLEHIAN AND MOSTAFA SATTARI

ABSTRACT. In this paper, we study the relationship between operator space norm and operator space numerical radius on the matrix space $\mathcal{M}_n(X)$, when X is a numerical radius operator space. Moreover, we establish several inequalities for operator space numerical radius and the maximal numerical radius norm of 2×2 operator matrices and their off-diagonal parts. One of our main results states that if $(X, (O_n))$ is an operator space, then

$$\begin{aligned} \frac{1}{2} \max (W_{\max}(x_1 + x_2), W_{\max}(x_1 - x_2)) \\ \leq W_{\max} \left(\begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix} \right) \\ \leq \frac{1}{2} (W_{\max}(x_1 + x_2) + W_{\max}(x_1 - x_2)) \end{aligned}$$

for all $x_1, x_2 \in \mathcal{M}_n(X)$.

1. INTRODUCTION

Let $\mathcal{B}(H)$ denote the C^* -algebra of bounded linear operators acting on a Hilbert space H . Let $\|a\|_n$ denote the operator norm and $w_n(a)$ stand for the numerical radius norm of an element a in the $n \times n$ matrix algebra $\mathcal{M}_n(\mathcal{B}(H))$ identifying with $\mathcal{B}(H^{(n)})$ in a natural way, where $H^{(n)}$ is the direct sum of n copy of H . Recall that the numerical radius norm of a is given by $w_n(a) = \sup\{|\langle ax, x \rangle| : x \in H^{(n)}, \|x\| = 1\}$. An (abstract) operator space is a complex linear space X together with a sequence of norms $O_n(\cdot)$ ($n = 1, 2, \dots$) defined on the $n \times n$ matrix space $\mathcal{M}_n(X)$ satisfying the following Ruan's axioms (cf. [3]):

$$O_{m+n} \left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right) = \max \{ O_m(x), O_n(y) \},$$

$$O_n(\alpha x \beta) \leq \|\alpha\| O_m(x) \|\beta\|.$$

for all $x \in \mathcal{M}_m(X)$, $y \in \mathcal{M}_n(X)$, $\alpha \in \mathcal{M}_{n,m}(\mathbb{C})$ and $\beta \in \mathcal{M}_{m,n}(\mathbb{C})$.

Ruan [13] proved that if $(X, (O_n))$ is an operator space, then there is a complete isometry ψ from X to $\mathcal{B}(H)$ for some Hilbert space H in the sense that $O_n(x) =$

2010 *Mathematics Subject Classification.* Primary 47A12; Secondary 46L15, 47A30, 47A63, 47L25.

Key words and phrases. Numerical radius operator space, operator space norm, maximal numerical radius norm, block matrix, operator space.

$\|\psi_n(x)\|_n$ for all $x \in \mathcal{M}_n(X)$ and $n \in \mathbb{N}$, where $\|\cdot\|_n$ is the usual operator norm of $\mathcal{M}_n(\mathcal{B}(H))$.

Itoh and Nagisa [7] introduced the notion of (abstract) numerical radius operator space (NROS), see also [8]. By a numerical radius operator space we mean a complex linear space X admitting a sequence of norms $W_n(\cdot)$ on $\mathcal{M}_n(X)$, $n \in \mathbb{N}$, for which

$$W_{m+n}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \max\{W_m(x), W_n(y)\}, \quad (1.1)$$

$$W_n(\alpha x \alpha^*) \leq \|\alpha\|^2 W_m(x). \quad (1.2)$$

for all $x \in \mathcal{M}_m(X)$, $y \in \mathcal{M}_n(X)$ and $\alpha \in \mathcal{M}_{n,m}(\mathbb{C})$, where α^* is the conjugate transpose of α .

They also showed that if $(X, (W_n))$ is a numerical radius operator space, then there is a W -complete isometry Φ from X to $\mathcal{B}(H)$ for some Hilbert space H in the sense that $W_n(x) = w_n(\Phi_n(x))$ for all $x \in \mathcal{M}_n(X)$ and $n \in \mathbb{N}$, where $w_n(\cdot)$ is the usual numerical radius norm on $\mathcal{B}(H^{(n)})$.

Having a look at the known equality

$$\frac{1}{2}\|x\| = w\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right), \quad x \in \mathcal{B}(H).$$

it is shown [7] that for a given numerical radius operator space $(X, (W_n))$ if one defines O_n ($n \in \mathbb{N}$) by

$$O_n(x) := 2W_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right), \quad x \in \mathcal{M}_n(X), \quad (1.3)$$

then X turns into an operator space. It is interesting to notice that if an operator space $(X, (O_n))$ is given, then there may be more than one operator space numerical radius (W_n) satisfying (1.3), [7]. For instance, consider the maximal numerical radius norm W_{\max} on an operator space $(X, (O_n))$, which is defined by

$$W_{\max}(x) = \frac{1}{2} \inf \|aa^* + b^*b\|, \quad \text{for } x \in \mathcal{M}_n(X),$$

where the infimum is taken over all decompositions $x = ayb$ with $O_r(y) = 1$, $a \in \mathcal{M}_{n,r}(\mathbb{C})$, $y \in \mathcal{M}_r(X)$, $b \in \mathcal{M}_{r,n}(\mathbb{C})$, $r \in \mathbb{N}$. It is proved in [7] that W_{\max} satisfies (1.1), (1.2) and (1.3).

There have been several generalizations of the usual numerical range in the last few decades. These concepts are useful in investigation of quantum error correction and perturbation theory (e.g., see [2, 4, 10, 11, 12] and references therein). Several mathematicians [5, 6, 9] established some interesting inequalities for the block matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and also its off-diagonal part, i.e. $\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}$. There are other papers involving numerical radius inequalities; cf. [1, 14]. In this paper, we obtain inequalities for $W_{2n}(\cdot)$ and W_{\max} of 2×2 block matrices with entries in appropriate matrix spaces similar to inequalities given in [5]. These inequalities include bounds for 2×2

block matrices. Furthermore, a generalization of a well known lemma given in [7] is established.

2. INEQUALITIES FOR OPERATOR SPACE NUMERICAL RADIUS AND THE MAXIMAL NUMERICAL RADIUS NORM

In this section, we provide an inequality between operator space norm and operator space numerical radius similar to the usual operator norm and the usual numerical radius norm. Also we apply it to give bounds for the off-diagonal part $\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$ of the 2×2 block matrix $\begin{bmatrix} z & x \\ y & w \end{bmatrix}$ defined on $\mathcal{M}_2(\mathcal{M}_n(X))$. First we fix our notation and terminology.

Given abstract numerical radius operator spaces (resp., operator spaces) X, Y and a linear map φ from X to Y , we define φ_n from $\mathcal{M}_n(X)$ to $\mathcal{M}_n(Y)$ by

$$\varphi_n([x_{ij}]) = [\varphi(x_{ij})], \quad [x_{ij}] \in \mathcal{M}_n(X).$$

We denote the numerical radius norm (resp., the norm) of $x = [x_{ij}] \in \mathcal{M}_n(X)$ by $W_n(x)$ (resp., $O_n(x)$) and the norm of φ_n by $W_n(\varphi_n) = \sup\{W_n(\varphi_n(x)) \mid x \in \mathcal{M}_n(X), W_n(x) \leq 1\}$ (resp., $O_n(\varphi_n) = \sup\{O_n(\varphi_n(x)) \mid x \in \mathcal{M}_n(X), O_n(x) \leq 1\}$). The W -completely bounded norm (resp., completely bounded norm) of φ is defined by

$$W(\varphi)_{cb} = \sup\{W_n(\varphi_n) \mid n \in \mathbb{N}\} \quad (\text{resp., } O(\varphi)_{cb} = \sup\{O_n(\varphi_n) \mid n \in \mathbb{N}\}).$$

We say φ is W -completely bounded (resp., completely bounded) if $W(\varphi)_{cb} < \infty$ (resp., $O(\varphi)_{cb} < \infty$) and also we call φ a W -complete isometry (resp., a complete isometry) if $W(\varphi_n(x)) = W_n(x)$ (resp., $O(\varphi_n(x)) = O_n(x)$) for each $x \in \mathcal{M}_n(X)$, $n \in \mathbb{N}$.

First of all we present a relation between $W_n(\cdot)$ and $O_n(\cdot)$.

Lemma 2.1. *If $(X, (W_n))$ is an NROS, then there is an operator space norm (O_n) on X such that*

$$\frac{1}{2} O_n(x) \leq W_n(x) \leq O_n(x).$$

for all $x \in \mathcal{M}_n(X)$ and $n \in \mathbb{N}$.

Proof. For given $(W_n(\cdot))$ and $x \in \mathcal{M}_n(X)$, we define $(O_n(\cdot))$ by

$$O_n(x) = 2W_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right).$$

Then there exists a complete and W -complete isometry Φ from X into $\mathcal{B}(H)$ [7]. As Φ is a complete isometry, we have $O_n(x) = \|\Phi_n(x)\|_n$. In addition, since Φ is a W -complete isometry, we have $W_n(x) = w_n(\Phi_n(x))$. Therefore,

$$W_n(x) = w_n(\Phi_n(x)) \leq \|\Phi_n(x)\|_n = O_n(x).$$

and

$$W_n(x) = w_n(\Phi_n(x)) \geq \frac{1}{2} \|\Phi_n(x)\|_n = \frac{1}{2} O_n(x).$$

□

The next result can be proved easily and we omit its proof.

Lemma 2.2. *If $(X, (W_n))$ is an NROS and $U \in \mathcal{M}_n$ is a unitary, then*

$$W_n(U^* x U) = W_n(x) \quad (2.1)$$

for any $x \in \mathcal{M}_n(X)$.

By a similar way, identity (2.1) is valid for W_{\max} . Also it should be mentioned here that $(O_n(\cdot))$ is unitarily invariant, i.e. $O_n(UxV) = O_n(x)$ for all unitary $U, V \in \mathcal{M}_n$ and $x \in \mathcal{M}_n(X)$.

Now, we use triangle inequality for $W_n(\cdot)$ and give upper and lower bounds for $W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right)$.

Lemma 2.3. *If $(X, (W_n))$ is an NROS, then*

$$\frac{1}{2} \max(O_n(x), O_n(y)) \leq W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) \leq \frac{1}{2} (O_n(x) + O_n(y))$$

for some operator space norm $(O_n(\cdot))$.

Proof. By (1.3), there is an operator space norm $(O_n(\cdot))$ on X such that

$$O_n(x) = 2W_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right).$$

First we prove the second inequality. Hence,

$$\begin{aligned} W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) &\leq W_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) + W_{2n}\left(\begin{bmatrix} 0 & 0 \\ y & 0 \end{bmatrix}\right) \\ &= \frac{1}{2} O_n(x) + W_{2n}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \\ &\leq \frac{1}{2} O_n(x) + W_{2n}\left(\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right) \quad (\text{by inequality (1.2)}) \\ &= \frac{1}{2} (O_n(x) + O_n(y)). \end{aligned}$$

To proving the first inequality, we use Ruan's axioms as follows.

$$\begin{aligned}
W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) &\geq \frac{1}{2} O_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \quad (\text{by Lemma (2.1)}) \\
&\geq \frac{1}{2} O_{2n} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) \\
&= \frac{1}{2} O_{2n} \left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right) = \frac{1}{2} O_n(x).
\end{aligned}$$

Similarly $W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \geq \frac{1}{2} O_n(y)$. □

Remark 2.4. Utilizing Lemma 2.1, the inequalities of Lemma 2.3 can be stated as follows:

$$\frac{1}{2} \max(W_n(x), W_n(y)) \leq W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \leq W_n(x) + W_n(y).$$

Now we are in a position to verify a general inequality for $W_n(\cdot)$, which contains some inequalities as special cases.

Theorem 2.5. *Let $(X, (W_n))$ be an NROS. Then for each $x, y \in \mathcal{M}_n(X)$ and $\alpha, \beta, \gamma, \delta \in \mathcal{M}_n(\mathbb{C})$*

$$W_n(\alpha x \beta \pm \gamma y \delta) \leq (\|\alpha\| \|\beta\| + \|\gamma\| \|\delta\|) \max(O_n(x), O_n(y)),$$

where $(O_n(\cdot))$ is a certain operator space norm.

Proof. Assume that $(O_n(\cdot))$ is defined by (1.3). Using the second inequality of Lemma 2.1, Ruan's axioms of operator spaces and the C^* -identity, we have

$$\begin{aligned}
W_n(\alpha x \beta + \gamma y \delta) &\leq O_n(\alpha x \beta + \gamma y \delta) = O_{2n} \left(\begin{bmatrix} \alpha & \gamma \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} \beta \\ \delta \end{bmatrix} \right) \\
&\leq \left\| \begin{bmatrix} \alpha & \gamma \end{bmatrix} \right\| O_{2n} \left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right) \left\| \begin{bmatrix} \beta \\ \delta \end{bmatrix} \right\| \\
&= \|\alpha \alpha^* + \gamma \gamma^*\|^{\frac{1}{2}} O_{2n} \left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right) \|\beta^* \beta + \delta^* \delta\|^{\frac{1}{2}} \\
&\leq \frac{1}{2} (\|\alpha \alpha^* + \gamma \gamma^*\| + \|\beta^* \beta + \delta^* \delta\|) O_{2n} \left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right) \\
&\leq \frac{1}{2} (\|\alpha\|^2 + \|\beta\|^2 + \|\gamma\|^2 + \|\delta\|^2) O_{2n} \left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right) \quad (2.2)
\end{aligned}$$

Let $t > 0$. Replace $\alpha, \beta, \gamma, \delta$ by $t\alpha, t^{-1}\beta, t\gamma, t^{-1}\delta$, respectively, in inequality (2.2) and use the following equality

$$\inf_{t>0} \frac{t^2 u + t^{-2} v}{2} = \sqrt{uv}$$

to get

$$W_n(\alpha x \beta + \gamma y \delta) \leq (\|\alpha\| \|\beta\| + \|\gamma\| \|\delta\|) \max(O_n(x), O_n(y)).$$

To complete the proof, it is sufficient to replace y by $-y$ in the above inequality. \square

Corollary 2.6. *If $(X, (W_n))$ is an NROS, then there exists an operator space norm $(O_n(\cdot))$ such that for any $x, y \in \mathcal{M}_n(X)$ and $\alpha, \beta \in \mathcal{M}_n(\mathbb{C})$, it holds that*

$$W_n(\alpha x \beta \pm \beta y \alpha) \leq 2\|\alpha\| \|\beta\| \max(O_n(x), O_n(y)). \quad (2.3)$$

In particular,

$$W_n(\alpha x \pm y \alpha) \leq 2\|\alpha\| \max(O_n(x), O_n(y)).$$

and

$$W_n(\alpha x \pm x \alpha) \leq 2\|\alpha\| O_n(x).$$

Proof. To show inequality (2.3), it is enough to take $\gamma = \beta$ and $\delta = \alpha$ in Theorem 2.5. The other inequalities follow immediately from inequality (2.3). \square

Corollary 2.7. *Suppose $(X, (W_n))$ is an NROS. Then there exists an operator space norm $(O_n(\cdot))$ such that for any $x, y \in \mathcal{M}_n(X)$ and $\alpha, \gamma \in \mathcal{M}_n(\mathbb{C})$, it holds that*

$$W_n(\alpha x \pm \gamma y) \leq (\|\alpha\| + \|\gamma\|) \max(O_n(x), O_n(y)).$$

In particular,

$$W_n(\alpha x \pm \gamma x) \leq (\|\alpha\| + \|\gamma\|) O_n(x).$$

Proof. The first inequality immediately follows from taking $\beta = \delta = I$ in Theorem 2.5, and for the second inequality it is sufficient to put $x = y$ in the first inequality. \square

Next we present more results for the operator space numerical radius of 2×2 off-diagonal block matrices. To do this, we need the following lemma.

Lemma 2.8. *Let $(X, (W_n))$ be an NROS. Then for each $x, y \in \mathcal{M}_n(X)$*

$$(a) \quad W_{2n} \left(\begin{bmatrix} 0 & x \\ e^{i\theta} y & 0 \end{bmatrix} \right) = W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \text{ for } \theta \in \mathbb{R},$$

$$(b) \quad W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) = W_{2n} \left(\begin{bmatrix} 0 & y \\ x & 0 \end{bmatrix} \right),$$

$$(c) \quad W_{2n} \left(\begin{bmatrix} x & y \\ y & x \end{bmatrix} \right) = \max(W_n(x+y), W_n(x-y)),$$

In particular,

$$W_{2n} \left(\begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix} \right) = W_n(y).$$

$$(d) \quad W_{2n} \left(\begin{bmatrix} y & -x \\ x & y \end{bmatrix} \right) = \max(W_n(x+iy), W_n(x-iy)).$$

Note that if $(X, (O_n))$ is an operator space, then all above statements hold for W_{\max} .

Proof. Parts (a) and (b) can be easily concluded by utilizing identity (2.1) to the matrix $\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$ and the unitary operators $\begin{bmatrix} I & 0 \\ 0 & e^{\frac{i\theta}{2}}I \end{bmatrix}$ and $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, respectively. Part (c) follows from applying identity (2.1) to the matrix $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$ and the unitary $\frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$. To verify part (d), first we use identity (2.1) to the matrix $\begin{bmatrix} iy & -x \\ x & iy \end{bmatrix}$ and the unitary $\frac{1}{\sqrt{2}} \begin{bmatrix} I & iI \\ iI & I \end{bmatrix}$ to get

$$W_{2n} \left(\begin{bmatrix} iy & -x \\ x & iy \end{bmatrix} \right) = \max(W_n(x+y), W_n(x-y)).$$

Taking $-iy$ instead of y in the above identity we reach part (d). \square

Our first main result is stated as follows.

Theorem 2.9. *Let $(X, (W_n))$ be an NROS and $x, y \in \mathcal{M}_n(X)$. Then*

$$W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \geq \frac{1}{2} \max(W_n(x+y), W_n(x-y))$$

and

$$W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \leq \frac{1}{2} (W_n(x+y) + W_n(x-y)).$$

Proof.

$$\begin{aligned} W_n(x+y) &= W_n \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \\ &\leq \left\| \begin{bmatrix} 1 & 1 \end{bmatrix} \right\|^2 W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \quad (\text{by inequality (1.2)}) \\ &= 2W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right). \end{aligned}$$

Hence,

$$\frac{1}{2} W_n(x+y) \leq W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right). \quad (2.4)$$

Replacing y by $-y$ in inequality (2.4), we get

$$\frac{1}{2} W_n(x-y) \leq W_{2n} \left(\begin{bmatrix} 0 & x \\ -y & 0 \end{bmatrix} \right) = W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \quad (\text{by Lemma 2.8 (a)}) \quad (2.5)$$

Now, the first inequality follows from inequalities (2.4) and (2.5). To prove the second inequality, we apply triangle inequality and Lemma 2.8 as follows:

$$\begin{aligned}
W_n(x+y) + W_n(x-y) &= W_{2n} \left(\begin{bmatrix} 0 & x+y \\ x+y & 0 \end{bmatrix} \right) + W_{2n} \left(\begin{bmatrix} 0 & x-y \\ x-y & 0 \end{bmatrix} \right) \\
&= W_{2n} \left(\begin{bmatrix} 0 & x+y \\ x+y & 0 \end{bmatrix} \right) + W_{2n} \left(\begin{bmatrix} 0 & x-y \\ y-x & 0 \end{bmatrix} \right) \\
&\quad \text{(by Lemma 2.8 (a) and (c))} \\
&\geq W_{2n} \left(\begin{bmatrix} 0 & x+y \\ x+y & 0 \end{bmatrix} + \begin{bmatrix} 0 & x-y \\ y-x & 0 \end{bmatrix} \right) \\
&= 2W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right).
\end{aligned}$$

□

Corollary 2.10. *If $(X, (W_n))$ is an NROS and $x, y \in \mathcal{M}_n(X)$, then*

$$\max(W_n(x), W_n(y)) \leq W_{2n} \left(\begin{bmatrix} 0 & x+y \\ x-y & 0 \end{bmatrix} \right) \leq W_n(x) + W_n(y).$$

Proof. It's enough to take $x+y$ and $x-y$ instead of x and y , respectively, in Theorem 2.9. □

Proposition 2.11. *Suppose $(X, (W_n))$ is an NROS and $x, y \in \mathcal{M}_n(X)$. Then*

$$W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \leq \min(W_n(x), W_n(y)) + \frac{\min(O_n(x+y), O_n(x-y))}{2}$$

for some operator space norm $(O_n(\cdot))$.

Proof. By Lemma 2.8 (a), (b) and identity (1.3), we get

$$\begin{aligned}
\frac{1}{2} O_n(x+y) + W_n(y) &= W_{2n} \left(\begin{bmatrix} 0 & x+y \\ 0 & 0 \end{bmatrix} \right) + W_{2n} \left(\begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix} \right) \\
&= W_{2n} \left(\begin{bmatrix} 0 & x+y \\ 0 & 0 \end{bmatrix} \right) + W_{2n} \left(\begin{bmatrix} 0 & -y \\ y & 0 \end{bmatrix} \right) \\
&\geq W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \text{ (by triangle inequality)} \quad (2.6)
\end{aligned}$$

Replacing y by $-y$ in inequality (2.6) and using Lemma 2.8 (a), we obtain

$$W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \leq \frac{1}{2} O_n(x-y) + W_n(y). \quad (2.7)$$

It follows from inequalities (2.6) and (2.7) that

$$W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \leq \frac{\min(O_n(x+y), O_n(x-y))}{2} + W_n(y). \quad (2.8)$$

Interchanging x and y in inequality (2.8) and using Lemma 2.8 (b), we get

$$W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) \leq \frac{\min(O_n(x+y), O_n(x-y))}{2} + W_n(x). \quad (2.9)$$

Now the result follows from inequalities (2.8) and (2.9). \square

Theorem 2.12. *Let $(X, (W_n))$ be an NROS and $x, y \in \mathcal{M}_n(X)$. Then*

$$W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) \geq \left| \frac{1}{2} \max(O_n(x+y), O_n(x-y)) - \min(W_n(x), W_n(y)) \right|,$$

and

$$W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) \geq \left| \max(W_n(x), W_n(y)) - \frac{1}{2} \min(O_n(x+y), O_n(x-y)) \right|.$$

Proof. Utilizing identity (1.3), Lemma 2.8 (a) and (c), we get

$$\begin{aligned} \frac{1}{2} O_n(x+y) &= W_{2n}\left(\begin{bmatrix} 0 & x+y \\ 0 & 0 \end{bmatrix}\right) = W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} + \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix}\right) \\ &\leq W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) + W_n(y). \end{aligned} \quad (2.10)$$

Replacing y by $-y$ in inequality (2.10) and using Lemma 2.8 (a) we have

$$\frac{1}{2} O_n(x-y) \leq W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) + W_n(y). \quad (2.11)$$

So, by inequalities (2.10) and (2.11)

$$\frac{1}{2} \max(O_n(x+y), O_n(x-y)) \leq W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) + W_n(y). \quad (2.12)$$

Interchanging x and y in inequality (2.12) and using Lemma 2.8 (b) we reach

$$\frac{1}{2} \max(O_n(x+y), O_n(x-y)) \leq W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) + W_n(x). \quad (2.13)$$

It follows from inequalities (2.12) and (2.13) that

$$\frac{1}{2} \max(O_n(x+y), O_n(x-y)) - \min(W_n(x), W_n(y)) \leq W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right). \quad (2.14)$$

On the other hand, by identity (1.3), we have

$$\begin{aligned} W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) + \frac{1}{2} O_n(x-y) &= W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) + W_{2n}\left(\begin{bmatrix} 0 & x-y \\ 0 & 0 \end{bmatrix}\right) \\ &\geq W_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} - \begin{bmatrix} 0 & x-y \\ 0 & 0 \end{bmatrix}\right) \\ &= W_{2n}\left(\begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix}\right) = W_n(y). \end{aligned} \quad (2.15)$$

Again, by replacing y by $-y$ in inequality (2.15) and using Lemma 2.8 (a), we get

$$W_n(y) \leq W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) + \frac{1}{2} O_n(x+y). \quad (2.16)$$

We therefore infer, by inequalities (2.15) and (2.16), that

$$W_n(y) \leq W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) + \frac{1}{2} \max(O_n(x+y), O_n(x-y)). \quad (2.17)$$

In inequality (2.17) we interchange x and y and use Lemma 2.8 (b) to get

$$W_n(x) \leq W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) + \frac{1}{2} \max(O_n(x+y), O_n(x-y)). \quad (2.18)$$

It follows from inequalities (2.17) and (2.18) that

$$-\left(\frac{1}{2} \max(O_n(x+y), O_n(x-y)) - \min(W_n(x), W_n(y)) \right) \leq W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right). \quad (2.19)$$

Thus the first desired inequality follows immediately from inequalities (2.14) and (2.19).

The other inequality is deduced by a similar argument. \square

In the sequel, we present some inequalities for W_{\max} having common nature to our earlier results. The next theorem is one of our main results.

Theorem 2.13. *Let $(X, (O_n))$ be an operator space. Then*

$$\begin{aligned} \frac{1}{2} \max(W_{\max}(x_1 + x_2), W_{\max}(x_1 - x_2)) &\leq W_{\max} \left(\begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix} \right) \\ &\leq \frac{1}{2} (W_{\max}(x_1 + x_2) + W_{\max}(x_1 - x_2)) \end{aligned}$$

for all $x_1, x_2 \in \mathcal{M}_n(X)$.

Proof. For the first inequality, let $\begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix} = ayb$, $O_r(y) = 1$, for $a \in \mathcal{M}_{n,r}(\mathbb{C})$, $y \in \mathcal{M}_r(X)$, $b \in \mathcal{M}_{r,n}(\mathbb{C})$ and $r \in \mathbb{N}$. So, we can write

$$x_1 + x_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} ayb \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We derive from the definition of $W_{\max}(x_1 + x_2)$ that

$$\begin{aligned} \frac{1}{2} W_{\max}(x_1 + x_2) &\leq \frac{1}{4} \left\| \begin{bmatrix} 1 & 1 \end{bmatrix} aa^* \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} b^*b \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| \\ &= \frac{1}{4} \left\| \begin{bmatrix} 1 & 1 \end{bmatrix} (aa^* + b^*b) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| \\ &\leq \frac{1}{2} \|aa^* + b^*b\|. \end{aligned}$$

whence

$$\frac{1}{2}W_{\max}(x_1 + x_2) \leq W_{\max}\left(\begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix}\right). \quad (2.20)$$

Replacing x_2 by $-x_2$ in inequality (2.20) and using Lemma 2.8 (a) for W_{\max} , we get

$$\frac{1}{2}W_{\max}(x_1 - x_2) \leq W_{\max}\left(\begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix}\right). \quad (2.21)$$

The first inequality now deduce from inequalities (2.20) and (2.21).

For the second inequality, it's sufficient to prove that

$$W_{\max}\left(\begin{bmatrix} 0 & x_1 + x_2 \\ x_1 - x_2 & 0 \end{bmatrix}\right) \leq W_{\max}(x_1) + W_{\max}(x_2).$$

For any $x_1, x_2 \in \mathcal{M}_n(X)$ and given $\epsilon > 0$, we may choose $a_i \in \mathcal{M}_{n,r}(\mathbb{C})$, $b_i \in \mathcal{M}_{r,n}(\mathbb{C})$, $y_i \in \mathcal{M}_r(X)$ with $O_r(y_i) = 1$ such that $x_i = a_i y_i b_i$ ($i = 1, 2$) and

$$W_{\max}(x_1) + \epsilon \geq \frac{1}{2}\|a_1 a_1^* + b_1^* b_1\|, \quad W_{\max}(x_2) + \epsilon \geq \frac{1}{2}\|a_2 a_2^* + b_2^* b_2\|.$$

Now we can write the following representation:

$$\begin{bmatrix} 0 & x_1 + x_2 \\ x_1 - x_2 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & a_2 \end{bmatrix} \begin{bmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & y_2 \end{bmatrix} \begin{bmatrix} 0 & b_1 \\ 0 & b_2 \\ b_1 & 0 \\ -b_2 & 0 \end{bmatrix}.$$

It follows that

$$\begin{aligned} & W_{\max}\left(\begin{bmatrix} 0 & x_1 + x_2 \\ x_1 - x_2 & 0 \end{bmatrix}\right) \\ & \leq \frac{1}{2} \left\| \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & a_2 \end{bmatrix} \begin{bmatrix} a_1^* & 0 \\ a_2^* & 0 \\ 0 & a_1^* \\ 0 & a_2^* \end{bmatrix} + \begin{bmatrix} 0 & 0 & b_1^* & -b_2^* \\ b_1^* & b_2^* & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & b_1 \\ 0 & b_2 \\ b_1 & 0 \\ -b_2 & 0 \end{bmatrix} \right\| \\ & = \frac{1}{2} \|a_1 a_1^* + a_2 a_2^* + b_1^* b_1 + b_2^* b_2\| \\ & \leq \frac{1}{2} \|a_1 a_1^* + b_1^* b_1\| + \frac{1}{2} \|a_2 a_2^* + b_2^* b_2\| \\ & \leq W_{\max}(x_1) + W_{\max}(x_2) + 2\epsilon. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we get the required inequality. \square

In the next result, other lower and upper bounds for W_{\max} are furnished.

Proposition 2.14. *Suppose $(X, (O_n))$ is an operator space. Then*

$$\frac{1}{2} \max(W_{\max}(x_1), W_{\max}(x_2)) \leq W_{\max}\left(\begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix}\right) \leq W_{\max}(x_1) + W_{\max}(x_2)$$

for $x_1, x_2 \in \mathcal{M}_n(X)$.

Proof. It turns out from inequalities (2.20) and (2.21) that

$$\begin{aligned} 2W_{\max}\left(\begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix}\right) &\geq \frac{1}{2}W_{\max}(x_1 + x_2) + \frac{1}{2}W_{\max}(x_1 - x_2) \\ &\geq \frac{1}{2}W_{\max}(x_1 + x_2 + x_1 - x_2) = W_{\max}(x_1). \end{aligned}$$

Therefore,

$$W_{\max}\left(\begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix}\right) \geq \frac{1}{2}W_{\max}(x_1). \quad (2.22)$$

In a similar manner,

$$W_{\max}\left(\begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix}\right) \geq \frac{1}{2}W_{\max}(x_2). \quad (2.23)$$

The first inequality follows immediately from (2.22) and (2.23). To get the second inequality assume $x_1, x_2 \in \mathcal{M}_n(X)$ and $\epsilon > 0$. we may select $a_i \in \mathcal{M}_{n,r}(\mathbb{C})$, $b_i \in \mathcal{M}_{r,n}(\mathbb{C})$, $y_i \in \mathcal{M}_r(X)$ with $x_i = a_i y_i b_i$ ($i = 1, 2$) and

$$W_{\max}(x_1) + \epsilon \geq \frac{1}{2}\|a_1 a_1^* + b_1^* b_1\|, \quad W_{\max}(x_2) + \epsilon \geq \frac{1}{2}\|a_2 a_2^* + b_2^* b_2\|.$$

The decomposition

$$\begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}$$

yields that

$$\begin{aligned} W_{\max}\left(\begin{bmatrix} 0 & x_1 \\ x_2 & 0 \end{bmatrix}\right) &\leq \frac{1}{2}\left\|\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}^* + \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}^*\begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}\right\| \\ &= \frac{1}{2}\left\|\begin{bmatrix} a_1 a_1^* + b_2^* b_2 & 0 \\ 0 & a_2 a_2^* + b_1^* b_1 \end{bmatrix}\right\| \\ &= \frac{1}{2}\max(\|a_1 a_1^* + b_2^* b_2\|, \|a_2 a_2^* + b_1^* b_1\|) \\ &\leq \frac{1}{2}\|a_1 a_1^* + b_2^* b_2 + a_2 a_2^* + b_1^* b_1\| \\ &\leq \frac{1}{2}\|a_1 a_1^* + b_1^* b_1\| + \frac{1}{2}\|a_2 a_2^* + b_2^* b_2\| \\ &\leq W_{\max}(x_1) + W_{\max}(x_2) + 2\epsilon. \end{aligned} \quad (2.24)$$

where inequality (2.24) follows from the fact that, if $A, B \in \mathcal{B}(H)$ are positive operator, then $\max(\|A\|, \|B\|) \leq \|A + B\|$. Now since $\epsilon > 0$ is arbitrary, we obtain the desired inequality. \square

3. UPPER AND LOWER BOUNDS OF 2×2 BLOCK MATRICES

In this section, first we present some pinching inequalities for W_n . Moreover, we provide different bounds for 2×2 block matrices of the form $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Some other related inequalities are also discussed.

Lemma 3.1. *Assume $(X, (W_n))$ is an NROS and $x, y, z, w \in \mathcal{M}_n(X)$. Then*

$$W_{2n} \left(\begin{bmatrix} x & 0 \\ 0 & w \end{bmatrix} \right) \leq W_{2n} \left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \right),$$

and

$$W_{2n} \left(\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix} \right) \leq W_{2n} \left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \right).$$

Proof. The first inequality can easily follow from $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, by considering unitary $U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, triangle inequality and identity (2.1) as

$$\begin{bmatrix} x & 0 \\ 0 & w \end{bmatrix} = \frac{A + U^*AU}{2}.$$

For the second inequality, we use

$$\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix} = \frac{A - U^*AU}{2}.$$

□

Proposition 3.2. *Let $(X, (W_n))$ be an NROS and $x, y \in \mathcal{M}_n(X)$. Then*

$$\max(W_n(x), W_n(y)) \leq W_{2n} \left(\begin{bmatrix} x & y \\ -y & -x \end{bmatrix} \right) \leq W_n(x) + W_n(y). \quad (3.1)$$

Proof. On making use of Lemma 3.1, we get

$$W_n(x) = W_{2n} \left(\begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \right) \leq W_{2n} \left(\begin{bmatrix} x & y \\ -y & -x \end{bmatrix} \right)$$

and

$$W_n(y) = W_{2n} \left(\begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \right) \leq W_{2n} \left(\begin{bmatrix} x & y \\ -y & -x \end{bmatrix} \right).$$

Therefore,

$$\max(W_n(x), W_n(y)) \leq W_{2n} \left(\begin{bmatrix} x & y \\ -y & -x \end{bmatrix} \right).$$

On the other hand, by employing triangle inequality, inequality (1.1), Lemma 2.8 (a) and (c), we have

$$\begin{aligned} W_{2n}\left(\begin{bmatrix} x & y \\ -y & -x \end{bmatrix}\right) &\leq W_{2n}\left(\begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix}\right) + W_{2n}\left(\begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix}\right) \\ &= W_n(x) + W_n(y). \end{aligned}$$

□

Remark 3.3. If we choose $y = x$ in inequality (3.1), then for $x \in \mathcal{M}_n(X)$

$$W_n(x) \leq W_{2n}\left(\begin{bmatrix} x & x \\ -x & -x \end{bmatrix}\right) \leq 2W_n(x). \quad (3.2)$$

Now we show that

$$W_{2n}\left(\begin{bmatrix} x & x \\ -x & -x \end{bmatrix}\right) = O_n(x), \quad x \in \mathcal{M}_n(X).$$

Using identities (2.1), (1.3) with the unitary $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$ we have

$$\begin{aligned} W_{2n}\left(\begin{bmatrix} x & x \\ -x & -x \end{bmatrix}\right) &= W_{2n}\left(\frac{1}{2} \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \begin{bmatrix} x & x \\ -x & -x \end{bmatrix} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}\right) \\ &= \frac{1}{2} W_{2n}\left(\begin{bmatrix} 0 & 4x \\ 0 & 0 \end{bmatrix}\right) = 2W_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = O_n(x). \end{aligned}$$

Based on the above identity, one can conclude that the inequalities of Lemma 2.1 and inequalities (3.2) are equivalent.

The next result provide a lower and upper bound for $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

Proposition 3.4. *Let $(X, (W_n))$ be an NROS and $x, y, z, w \in \mathcal{M}_n(X)$. Then*

$$W_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \geq \max\left(W_n(x), W_n(w), \frac{W_n(y)}{2}, \frac{W_n(z)}{2}\right)$$

and

$$W_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \leq W_n(x) + W_n(y) + W_n(z) + W_n(w).$$

Proof. Utilizing Lemma 3.1 and the first inequality of Remark 2.4, we derive

$$\begin{aligned} W_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) &\geq \max\left(W_{2n}\left(\begin{bmatrix} x & 0 \\ 0 & w \end{bmatrix}\right), W_{2n}\left(\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}\right)\right) \\ &\geq \max\left(\max(W_n(x), W_n(w)), \max\left(\frac{W_n(y)}{2}, \frac{W_n(z)}{2}\right)\right) \\ &= \max\left(W_n(x), W_n(w), \frac{W_n(y)}{2}, \frac{W_n(z)}{2}\right). \end{aligned}$$

To verify the other inequality first we present an upper bound to the matrix $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$.

To achieve this, we use the triangle inequality as follows:

$$\begin{aligned}
W_{2n}\left(\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}\right) &\leq W_{2n}\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right) + W_{2n}\left(\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right) \\
&= W_n(x) + \frac{1}{2} O_n(y) \\
&\quad \text{(by inequality (1.1) and identity (1.3))} \\
&\leq W_n(x) + W_n(y) \quad \text{(by Lemma 2.1)} \tag{3.3}
\end{aligned}$$

For the general case consider unitary $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. We infer by identity (2.1) that

$$\begin{aligned}
W_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) &\leq W_{2n}\left(\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}\right) + W_{2n}\left(\begin{bmatrix} 0 & 0 \\ z & w \end{bmatrix}\right) \\
&= W_{2n}\left(\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}\right) + W_{2n}\left(U^* \begin{bmatrix} w & z \\ 0 & 0 \end{bmatrix} U\right) \\
&= W_{2n}\left(\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}\right) + W_{2n}\left(\begin{bmatrix} w & z \\ 0 & 0 \end{bmatrix}\right) \\
&\leq W_n(x) + W_n(y) + W_n(z) + W_n(w). \\
&\quad \text{(by inequality (3.3))}
\end{aligned}$$

□

Another upper bound for $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ can be stated as follows.

Theorem 3.5. *Let $(X, (W_n))$ be an NROS and $x, y, z, w \in \mathcal{M}_n(X)$. Then*

$$\begin{aligned}
W_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) &\leq \max\left(\frac{W_n(x + w + i(y - z))}{2}, \frac{W_n(x + w - i(y - z))}{2}\right) \\
&\quad + \frac{W_n(w - x) + W_n(y + z)}{2}.
\end{aligned}$$

Proof. Applying identity (2.1) to the matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and unitary $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix}$, we have

$$\begin{aligned}
W_{2n} \left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \right) &= W_{2n} \left(U^* \begin{bmatrix} x & y \\ z & w \end{bmatrix} U \right) \\
&= \frac{1}{2} W_{2n} \left(\begin{bmatrix} x+y+z+w & -x+y-z+w \\ -x-y+z+w & x-y-z+w \end{bmatrix} \right) \\
&= \frac{1}{2} W_{2n} \left(\begin{bmatrix} x+w & y-z \\ z-y & x+w \end{bmatrix} + \begin{bmatrix} y+z & w-x \\ w-x & -z-y \end{bmatrix} \right) \\
&\leq \frac{1}{2} \left(W_{2n} \left(\begin{bmatrix} x+w & y-z \\ z-y & x+w \end{bmatrix} \right) + W_{2n} \left(\begin{bmatrix} y+z & w-x \\ w-x & -z-y \end{bmatrix} \right) \right) \\
&\leq \frac{1}{2} \left(\max(W_n(x+w+i(y-z)), W_n(x+w-i(y-z))) \right. \\
&\quad \left. + W_n(w-x) + W_n(y+z) \right). \quad (\text{by Lemma 2.8 (c), (d)})
\end{aligned} \tag{3.4}$$

□

Remark 3.6. Suppose $(X, (W_n))$ is an NROS and $x, y, z, w \in \mathcal{M}_n(X)$. Then

$$W_{2n} \left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \right) \leq \max(W_n(x), W_n(w)) + \frac{W_n(y+z) + W_n(y-z)}{2}.$$

Proof. According to identity (3.4), we can write

$$\begin{aligned}
W_{2n} \left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \right) &= \frac{1}{2} W_{2n} \left(\begin{bmatrix} x+w & w-x \\ w-x & x+w \end{bmatrix} + \begin{bmatrix} y+z & y-z \\ z-y & -z-y \end{bmatrix} \right) \\
&\leq \frac{1}{2} \left(W_{2n} \left(\begin{bmatrix} x+w & w-x \\ w-x & x+w \end{bmatrix} \right) + W_{2n} \left(\begin{bmatrix} y+z & y-z \\ z-y & -z-y \end{bmatrix} \right) \right) \\
&\leq \max(W_n(x), W_n(w)) + \frac{W_n(y+z) + W_n(y-z)}{2}. \\
&\quad (\text{by Lemma 2.8 (c)})
\end{aligned}$$

□

The last result in this section is a generalization of a well known Lemma in [7].

Proposition 3.7. *Suppose $(X, (W_n))$ be an NROS. If $f \in \mathcal{M}_n(X)^*$ and $W^*(f) \leq 1$, then there exists a state P_0 on $\mathcal{M}_n(\mathbb{C})$ such that*

$$|f(\alpha x \beta^* \pm \beta y \alpha^*)| \leq 2P_0(\alpha \alpha^*)^{\frac{1}{2}} P_0(\beta \beta^*)^{\frac{1}{2}} W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right),$$

for all $\alpha, \beta \in \mathcal{M}_{n,r}(\mathbb{C})$, $x, y \in \mathcal{M}_r(X)$, $r \in \mathbb{N}$.

In addition,

$$|f(\alpha x \beta \pm \gamma y \delta)| \leq \left(P_0(\alpha \alpha^*)^{\frac{1}{2}} P_0(\beta^* \beta)^{\frac{1}{2}} + P_0(\gamma \gamma^*)^{\frac{1}{2}} P_0(\delta^* \delta)^{\frac{1}{2}} \right) O_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \quad (3.5)$$

for all $\alpha, \gamma \in \mathcal{M}_{n,r}(\mathbb{C})$, $x, y \in \mathcal{M}_r(X)$, $\beta, \delta \in \mathcal{M}_{r,n}(\mathbb{C})$, $r \in \mathbb{N}$, where $W^*(f) = \sup\{|f(x)| : x \in \mathcal{M}_n(X), W_n(x) \leq 1\}$.

Proof. It is proved [7] under the same hypothesis that

$$\left| f(\alpha x \alpha^*) \right| \leq P_0(\alpha \alpha^*) W_n(x) \quad (3.6)$$

$$\left| f(\alpha x \beta) \right| \leq 2 P_0(\alpha \alpha^*)^{\frac{1}{2}} P_0(\beta^* \beta)^{\frac{1}{2}} W_{2n} \left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) \quad (3.7)$$

Now by inequality (3.6), we derive

$$\begin{aligned} |f(\alpha x \beta^* + \beta y \alpha^*)| &= \left| f \left(\begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \end{bmatrix}^* \right) \right| \\ &\leq P_0(\alpha \alpha^* + \beta \beta^*) W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right). \end{aligned}$$

Let $t > 0$ and replace α and β by $t\alpha$ and $\frac{1}{t}\beta$, respectively. Then the equality

$$\inf_{t>0} \{t^2 P_0(\alpha \alpha^*) + t^{-2} P_0(\beta \beta^*)\} = 2 P_0(\alpha \alpha^*)^{\frac{1}{2}} P_0(\beta \beta^*)^{\frac{1}{2}} \quad (3.8)$$

ensures

$$|f(\alpha x \beta^* + \beta y \alpha^*)| \leq 2 P_0(\alpha \alpha^*)^{\frac{1}{2}} P_0(\beta \beta^*)^{\frac{1}{2}} W_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right).$$

Replace y by $-y$ in the above inequality and use Lemma 2.8 (a) to deduce the first inequality of the proposition.

To verify inequality (3.5), we apply inequality (3.7) as follows:

$$\begin{aligned} |f(\alpha x \beta + \gamma y \delta)| &= \left| f \left(\begin{bmatrix} \alpha & \gamma \end{bmatrix} \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \begin{bmatrix} \delta \\ \beta \end{bmatrix} \right) \right| \\ &\leq P_0 \left(\begin{bmatrix} \alpha & \gamma \end{bmatrix} \begin{bmatrix} \alpha & \gamma \end{bmatrix}^* \right)^{\frac{1}{2}} P_0 \left(\begin{bmatrix} \delta \\ \beta \end{bmatrix}^* \begin{bmatrix} \delta \\ \beta \end{bmatrix} \right)^{\frac{1}{2}} O_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \\ &\quad \text{(by inequality (3.7) and identity (1.3))} \\ &= P_0(\alpha \alpha^* + \gamma \gamma^*)^{\frac{1}{2}} P_0(\beta^* \beta + \delta^* \delta)^{\frac{1}{2}} O_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) \\ &\leq \frac{1}{2} (P_0(\alpha \alpha^* + \gamma \gamma^*) + P_0(\beta^* \beta + \delta^* \delta)) O_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right). \\ &\quad \text{(by the arithmetic-geometric mean inequality)} \end{aligned}$$

If we replace $\alpha, \beta, \gamma, \delta$ by $t\alpha, t^{-1}\beta, t\gamma, t^{-1}\delta$, respectively, in the above inequality, then from equality (3.8) we get

$$|f(\alpha x\beta + \gamma y\delta)| \leq \left(P_0(\alpha\alpha^*)^{\frac{1}{2}} P_0(\beta^*\beta)^{\frac{1}{2}} + P_0(\gamma\gamma^*)^{\frac{1}{2}} P_0(\delta^*\delta)^{\frac{1}{2}} \right) O_{2n} \left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right). \quad (3.9)$$

Taking $-y$ instead of y in inequality (3.9) and using Lemma 2.8 (a), we reach inequality (3.5). \square

Noting that by letting $y = 0, \gamma = \delta = 0$ in inequality (3.5) and applying the first inequality of Lemma 2.1, we obtain inequality (3.7).

REFERENCES

1. A. Abu-Omar and F. Kittaneh, *Estimates for the numerical radius and the spectral radius of the Frobenius companion matrix and bounds for the zeros of polynomials*, Ann. Funct. Anal. **5** (2014), no. 1, 56–62.
2. M.D. Choi, D.W. Kribs and K. Życzkowski, *Quantum error correcting codes from the compression formalism*, Rep. Math. Phys. **58** (2006), 77–91.
3. E.G. Effros and Z.-J. Ruan, *Operator spaces*, London Mathematical Society Monographs. New Series, 23. The Clarendon Press, Oxford University Press, New York, 2000.
4. K.E. Gustafson and D.K.M. Rao, *Numerical range*, The field of values of linear operators and matrices. Universitext. Springer-Verlag, New York, 1997.
5. O. Hirzallah, F. Kittaneh and K. Shebrawi, *Numerical radius inequalities for certain 2×2 operator matrices*, Integral Equations Operator Theory **71** (2011), no. 1, 129–147.
6. O. Hirzallah, F. Kittaneh and K. Shebrawi, *Numerical radius inequalities for commutators of Hilbert space operators*, Numer. Funct. Anal. Optim. **32** (2011), 739–749.
7. T. Itoh and M. Nagisa, *Numerical radius norms on operator spaces*, J. London. Math. Soc. **74** (2006), 154–166.
8. T. Itoh and M. Nagisa, *Numerical radius Haagerup norm and square factorization through Hilbert spaces*, J. Math. Soc. Japan **58** (2006), no. 2, 363–377.
9. F. Kittaneh, M.S. Moslehian and T. Yamazaki, *Cartesian decomposition and numerical radius inequalities*, Linear Algebra Appl. **471** (2015), 46–53.
10. C.K. Li and Y.T. Poon, *Generalized numerical ranges and quantum error correction*, J. Operator Theory. **66** (2011), no. 2, 335–351.
11. C.K. Li, Y.T. Poon and N.-S. Sze, *Higher rank numerical ranges and low rank perturbations of quantum channels*, J. Math. Anal. Appl. **348** (2008), no. 2, 843–855.
12. Z. Puchała, J.A. Miszczyk, P. Gawron, C.F. Dunkl, J.A. Holbrook and K. Życzkowski, *Restricted numerical shadow and the geometry of quantum entanglement*, J. Phys. A **45** (2012), no. 41, 415309, 28 pp.
13. Z.-J. Ruan, *subspaces of C^* -algebras*, J. Funct. Anal. **76** (1988), 217–230.
14. M. Sattari, M.S. Moslehian and T. Yamazaki, *Some generalized numerical radius inequalities for Hilbert space operators*, Linear Algebra Appl. **470** (2015), no. 1, 216–227.

DEPARTMENT OF PURE MATHEMATICS, CENTER OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES (CEAAS), FERDOWSI UNIVERSITY OF MASHHAD, P. O. BOX 1159, MASHHAD 91775, IRAN

E-mail address: moslehian@um.ac.ir

E-mail address: msattari.b@gmail.com